Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XI. NOVEMBER, 1904. No. 11.

THE SUBTRACTION GROUPS.

By G. A. MILLER.

In the preceding volume of this journal* we considered the groups of the figures which are studied in elementary geometry and found that all the groups of the plane figures are of the dihedral rotation type and that every dihedral rotation group, with the exception of the one of order 4, is the group of movements of some regular polygon. In the present article we shall prove that these same groups are the groups of subtraction when the remainders are taken with respect to a modulus.

Although we shall deal with analytic concepts yet it seems best to make extensive use of geometric considerations. Let $x_1, x_2, \ldots\ldots, x_l$ denote any set of real integers and suppose that they are represented in the usual manner by points on a line. The numbers $n-x_1, n-x_2, \ldots\ldots, n-x_l$ (where n is any real number) correspond to the points obtained by reflecting the points corresponding to $x_1, x_2, \ldots\ldots, x_l$ on the point midway between 0 and n. Since reflection is an operation of period two subtraction from a single number is an operation of period two.

Instead of considering the remainders obtained by subtracting different numbers from the same number it seems desirable to consider the remainders when a given number is subtracted from a set of different numbers; *e. g.* the numbers $x_1-n, x_2-n, \ldots\ldots, x_l-n$. These l distinct operations of period two are in general, not commutative; for if we first subtract n from x_1 and then the resulting number from x_2 we obtain $n+x_2-x_1$, while $n+x_1-x_2$ is obtained when the order of these operations is reversed. Similarly, it may be observed geometrically that reflection with respect to distinct points is non-commutative. The main object of this paper is the study of the group generated by these l non-commutative operations. For simplicity, it will first be assumed that $l=2$.

---

*Vol. 10, 1903, p. 215.

Let s_1 represent the reflection corresponding to x_1-n while s_2 represents that corresponding to x_2-n. The two reflections $s_1 s_2$ are equal to the translation $n+x_2-x_1$. This translation has a finite period with respect to any finite integer (m) as modulus. This period is clearly m/d, where d is the highest common factor of x_2-x_1 and m. Since any two operators of period two generate the dihedral rotation group whose order is twice the order of the product of these operators,* it follows that s_1 and s_2 generate the dihedral rotation group of order $2m/d$.

To illustrate we may let s_1, s_2 represent respectively the taking of the complement and the supplement of an angle. In this case $m=360°$ and $d=90°$. Hence s_1, s_2 generate the dihedral rotation group of order 8, or the group of movements of the square. The eight operations of this group consist of the four rotations through 0°, 90°, 180°, and 270°, respectively, and the four reflections on the lines $x=0$, $x-y=0$, $y=0$, $x+y=0$, respectively.† It may be observed that the group is independent of the values of n. When n is a multiple of 45° the eight operations of the group give rise to only four distinct angles; viz. those which correspond to the four rotations.

These special values of n present a very instructive example in the application of group theory. Although an angle assumes all its possible values under the subgroup of rotations, yet each reflection transforms the totality of the angles in a different manner from these rotations and hence the group does not reduce for these special values of n. It should also be observed that the subgroup which leaves an angle invariant when n has one of these special values is not generally invariant.

It will soon be proved that there are $2m/d$ special values of n for which the subtraction group transforms n into less than $2m/d$ distinct values. Each of these special values is transformed into m/d distinct values. Hence the $2m/d$ special values may always be divided into two distinct sets such that each number of a set is transformed into every other number of the set by means of the cyclic subgroup of order m/d. The truth of these statements results readily from the following considerations. The two reflections s_1 s_2 are equivalent to a translation through the distance x_2-x_1 independently of the value of n. Hence the group which includes s_1, s_2 must always transform n into at least m/d distinct values. This is accomplished by the cyclic subgroup of order m/d. If the entire group transforms n into any other value it must transform it into $2m/d$ distinct values. From this it follows that the necessary and sufficient condition that there are only m/d distinct values of n under this group is that a reflection transforms n into the same value as a translation (or rotation), or that

---

Bulletin of the American Mathematical Society, Vol. 7, 1901, p. 424.

†Davis, *Nebraska University Studies*, Vol. 4, 1904, p. 231.

$$x_1-n\equiv n+k(x_2-x_1) \quad \text{mod } m.$$

Hence
$$n\equiv\frac{x_1}{2}+k\frac{x_1-x_2}{2} \quad \text{mod } m/2.$$

That is, n has $2m/d$ distinct values. These may always be represented as the vertices of a regular polygon of $2m/d$ sides, as will appear more clearly from what follows.

The preceding results may readily be extended so as to apply to the case when l has any arbitrary value. If d represents the highest common factor of m and all the differences of the form $x_\alpha - x_\beta$, the subtraction group generated by the reflections corresponding to the operations x_1-n, x_2-n, ..., x_l-n is again the dihedral rotation group of order $2m/d$. This results almost directly from the preceding considerations and the following elementary facts: (1) Any finite number of rational translations taken with respect to a rational modulus generate a cyclic group whose order is the quotient obtained by dividing the rational modulus by the largest number which is contained an integral number of times in all the translations and the modulus. This cyclic group may be generated by translations (or rotations*) which are equal to this largest divisor leading to integral quotients. (2) Any reflection followed by a translation is a reflection on the point obtained by translating the original point of reflection through half the original translation. That is, a translation followed by a reflection, or vice versa, is an operation of period two, or any translation whatever is transformed into its inverse by any reflection.

From these general considerations it follows that the subtraction group for an arbitrary value of l is just as elementary as when $l=2$, and that the number of special values of n is still equal to the order of the corresponding group and consist of two sets such that all the numbers of a set are obtained from any number of the set by the operations of the group. It is also clear that x_1, x_2, ..., x_l, m could be any rational numbers instead of being integral but this is only an apparent generalization and hence nothing would have been gained by the assumption.

When x_1, x_2, ..., x_l, m represent complex integers while n is any ordinary complex number the preceding considerations remain practically unchanged. For the sake of simplicity we shall again consider the case when $l=2$ by itself. Let $x_1=a_1+b_1i$, $x_2=a_2+b_2i$, $m=m_1+m_2i$, $n=x_0+y_0i$. The real parts of x_1-n, x_2-n are obtained by reflecting the real part of n on the lines $x=a_1/2$, $x=a_2/2$, respectively, while the coefficients of i in these remainders are obtained by reflecting y_0 on the lines $y=b_1/2$, $y=b_2/2$, respectively. The order of the translation generated by the two reflections x_1-n, x_2-n is the least common multiple of m_1/d_1, m_2/d_2, where d_1 is the highest common factor of a_2-a_1 and m_1 while d_2 has the same meaning with respect to b_2-b_1 and m_2. If m' represents these least common multiples the group generated by the two reflections x_1-n, x_2-n is the dihedral rotation group of order $2m'$.

---

*When the group is represented on a circle, the translations are replaced by rotations. This is sometimes most convenient.

The generalization of the case when l is arbitrary is so similar to the case considered above that it seems unnecessary to give details. The only important difference consists of the fact that m' represents the least common multiple of of m_1/d_1, m_2/d_2, where d_1 and d_2 are the highest common factors, respectively, of all such difference as $a_\alpha-a_\beta$ and $b_\alpha-b_\beta$. The numbers of the special values of n is again equal to the order of the corresponding group. It is clear that every dihedral group (including the four group) may be represented in an infinite number of ways as a subtraction group. The stress should, however, not be laid upon the fact that subtraction furnishes such interesting illustrations of this important system of groups, but rather upon the fact that these groups give a deeper and far reaching meaning to the fundamental operation of subtraction.

STANFORD UNIVERSITY, *September, 1904.*

THE SINKING-FUND OF THE UNITED STATES.

By G. B. M. ZERR.

The public debt of the United States is being paid by the sinking-fund in the following manner. During each fiscal year a sum is paid equal to one per cent. of the principal of the current debt, plus a sum equal to the interest on the part of the debt already paid at the rate of interest the debt bears. If such a sinking-fund had been operated under the same law from the beginning, how long would it require to pay the public debt, if the rate of interest the debt draws is four per cent. per annum?

A very excellent solution of the above problem is given in the *Mathematical Magazine* for September, 1904, by Theodore L. DeLand, who employs the Calculus of Finite Differences.

As the great debt of the United States will fall due in a few years, and, as its payment, then, will have to be met by a long-time loan at a different rate, which will change the present sinking-fund, we believe that a simple algebraic solution of this national problem will be interesting to the readers of the MONTHLY.

Let p=principal of the public debt at the beginning; r=.01, the rate per annum on the current principal; R=.04, the rate of interest the debt draws per annum; n=number of years required to pay the debt.

Then rp=first payment;

$p(1-r)$=unpaid part of debt after first payment;

$rp(1-r)+Rrp=pr(1-r+R)$=second payment;